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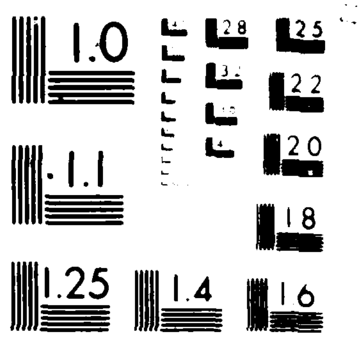
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20 ABSTRACT

Consider the canonical form MANOVA setup with $X: n \times p = (X_1', X_2', X_3')' = (M_1', M_2', 0)' + E, X_i: n_i \times p$, $i = 1, 2, 3$, $M_i: n_i \times p$, $i = 1, 2$, $n_1 + n_2 + n_3 = n$, $n_3 \geq p$, where E is a random error matrix with location 0 and unknown

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scale matrix $\Sigma > 0$(p.d.). Assume, unlike in the usual sense, that M_1 is random with location 0 and scale matrix $\sigma_1^2\Sigma$, M_2 is either fixed or random with location 0 and a different scale matrix $\sigma_2^2\Sigma$, σ_1^2, σ_2^2 being unknown. For testing H_0: $\sigma_1^2 = 0$ versus H_1: $\sigma_1^2 > 0$ under a left orthogonally invariant distribution of X, it is shown that when either $n_2 = 0$ or M_2 fixed if $n_2 > 0$ the trace test of Pillai (1955) is UMPI if $\min(n_1,p) = 1$ and LBI if $\min(n_1,p) > 1$. The test is null, nonnull and optimality robust (Kariya and Sinha (1985)). However, such a result does not hold if $n_2 > 0$ and M_2 random.

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ABSTRACT

Consider the canonical form MANOVA setup with $X: n\times p = (X_1', X_2', X_3')' = (M_1', M_2', 0)' + E, X_i: n_i \times p$, $i = 1, 2, 3$, $M_i: n_i \times p$, $i = 1, 2$, $n_1 + n_2 + n_3 = n$, $n_3 \geq p$, where E is a random error matrix with location 0 and unknown scale matrix $\Sigma > 0$(p.d.). Assume, unlike in the usual sense, that M_1 is random with location 0 and scale matrix $\sigma_1^2\Sigma$, M_2 is either fixed or random with location 0 and a different scale matrix $\sigma_2^2\Sigma$, σ_1^2, σ_2^2 being unknown. For testing $H_0: \sigma_1^2 = 0$ versus $H_1: \sigma_1^2 > 0$ under a left orthogonally invariant distribution of X, it is shown that when either $n_2 = 0$ or M_2 fixed if $n_2 > 0$ the trace test of Pillai (1955) is UMPI if $\min(n_1,p) = 1$ and LBI if $\min(n_1,p) > 1$. The test is null, nonnull and optimality robust (Kariya and Sinha (1985)). However, such a result does not hold if $n_2 > 0$ and M_2 random.

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1. INTRODUCTION

Tl usual MANOVA model in the canonical form consists of an nxp random data matrix X decomposed as $X = (X_1', X_2', X_3')'$ with $X_i: n_i xp$, i = 1, 2, 3, $n_1 + n_2 + n_3 = n$, $n_3 \geq p$, following the structure

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ 0 \end{bmatrix} + E. \tag{1.1}$$

Here $M_i: n_i xp$ is the mean matrix of X_i, i = 1, 2, and E: nxp is the random error matrix. Under the distributional assumption $E \sim N(0, I_n \otimes \Sigma)$ for some unknown p.d. pxp matrix Σ, many tests of the MANOVA hypothesis $H_0: M_1 = 0$ versus $H_1: M_1 \neq 0$ are well known, e.g. the likelihood ratio test, Roy's maximum root test, Lawley-Hotelling's trace test and Pillai's trace test. All these tests ignore X_2 and are functions of $X_1(X_3'X_3)^{-1}X_1'$ (vide Anderson (1984)). Moreover, the trace test of Pillai (1955) is known to be LBI in general (Schwartz (1967)) and UMPI if $\min(p, n_1) = 1$ (Lehmann (1959)). On the other hand, if M_1 and M_2 are assumed to be independent normal with zero mean and dispersion $\sigma_1^2\Sigma$ and $\sigma_2^2\Sigma$ respectively, Roy and Gnanadesikan (1959) considered the problem of testing $H_0: \sigma_1^2 = 0$ versus $H_1: \sigma_1^2 > 0$ and proposed the maximum root test, $\lambda_{max}(X_1(X_3'X_3)^{-1}X_1')$. See also Roy and Cobb (1960) for some related results. However, so far no optimum test is known.

It is the object of this paper to derive an optimum invariant test for testing $H_0: \sigma_1^2 = 0$ versus $H_1: \sigma_1^2 > 0$ under the model

$$X \sim f(x|\sigma_1^2, M_2, \Sigma) = |\Sigma|^{-n/2}(1+\sigma_1^2)^{-n_1/2} \tag{1.2}$$

$$q(\Sigma^{-1}(X_1'X_1/(1+\sigma_1^2) + (X_2-M_2)'(X_2-M_2) + X_3'X_3))$$

for some $q \in Q$. Here Q is the class of functions from the set of pxp matrices into $[0,\infty)$ such that $q \in Q$ satisfies

$$\int_{R^{np}} q(X'X)dX = 1, \tag{1.3}$$

$$\int_{G\ell(p)} \int_{R^{n_2p}} q(AA' + F'F)|AA'|^{(n_1+n_3-p)/2} dFdA < \infty$$

and

$$q(BV) = q(VB) \text{ for all } V \in \overline{\mathcal{L}}(p) \text{ and } B \in G\ell(p) \tag{1.4}$$

where F is a matrix of order $n_2$xp with elements in R^{n_2p}, dF is Lebesgue over R^{n_2p} and $\overline{\mathcal{L}}(p)$ is the set of pxp nonnegative definite matrices. The model (1.2) corresponds to (1.1) with only M_1 as random. This can be thought of as a mixed MANOVA model for $n_2 > 0$ and a random MANOVA model for $n_2 = 0$. Of course, unlike in previous papers, the normality of X has been replaced by a very general left orthogonally invariant distribution. We show that whatever be $q \in Q$, the trace test of Pillai (1955) is UMPI if $\min(n_1,p) = 1$ and LBI otherwise. In particular, for $n_2 = 0$ which makes the model (1.2) comparable to Roy and Gnanadesikan's (1959), the trace test is superior to the invariant maximum root test. Under normality of X, it is mentioned in Lehmann (1959, page 344) that when $n_2 = 0$ and $n_1 = n_2 = 1$, there exists a UMPI test under the group G_T of all pxp nonsingular lower triangular matrices with positive diagonal elements. This test is based on X_{11}^2/X_{31}^2 and, therefore, not very appealing due to its asymmetry. Here

X_{11} and X_{31} are the first components in the vectors $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_3$ respectively. It seems to us that for the above problem the group $G\ell(p)$ rather than G_T is the right group to use. For some discussion on properties of $q \in Q$, we refer to Kariya (1981).

The optimum invariant trace test is shown to be null, nonnull and optimality robust (vide Kariya and Sinha (1985)). It is interesting to compare our results with those of Kariya (1981) and Kariya and Sinha (1985) who proved similar results under the fixed effects MANOVA model (i.e. M_1, M_2 fixed matrices), Kariya (1981) requiring $q \in Q$ to be convex for the UMPI property to hold when $\min(n_2,p) = 1$, while Kariya and Sinha (1985) restricting q to belong to the class of ellistically symmetric distributions and satisfying some other conditions for the LBI property to hold when $\min(n_1,p) > 1$. However we do not impose any condition on q other than the integrability condition (1.3) and the condition (1.4). Moreover, our proof of the LBI property of the trace test for $\min(n_1,p) > 1$ is extremely simple due to the nature of the model (1.2). We refer to Schwartz (1967) and Kariya and Sinha (1985) for the LBI property of the trace test under fixed effects MANOVA model for normal q and elliptically symmetric q respectively.

If $n_2 > 0$ and M_2 random with mean zero and scale matrix $\sigma_2^2\Sigma$ so that the distribution of X follows

$$X \sim f(x|\sigma_1^2,\sigma_2^2,\Sigma) = |\Sigma|^{-n/2}(1+\sigma_1^2)^{-n_1/2}(1+\sigma_2^2)^{-n_2/2} \qquad (1.5)$$

$$q(\Sigma^{-1}(X_1'X_1/(1+\sigma_1^2) + X_2'X_2/(1+\sigma_2^2) + X_3'X_3))$$

a difficulty in the derivation of an optimum invariant test is pointed out.

2. ROBUST OPTIMUM INVARIANT TEST

Consider the model (1.2) and the problem of testing H_0: $\sigma_1^2 = 0$ versus H_1: $\sigma_1^2 > 0$ where $M_2 \in R^{n_2p}$ and $\Sigma > 0$ are unknown. It is easy to see that the problem is left invariant under the group $G = G\ell(p) \times R^{n_2p}$ acting on X and (M_2,Σ,σ_1^2) as

$$gX = (X_1A', X_2A' + F, X_3A') \tag{2.1}$$

and

$$g(M_2,\Sigma,\sigma_1^2) = (M_2A' + F, A\Sigma A', \sigma_1^2)$$

where $g = (A,F) \in G$, $A \in G\ell(p)$, $F \in R^{n_2 \times p}$. Here $G\ell(p)$ is the group of pxp nonsingular matrices and $R^{n_2 \times p}$ is the (additive) group of matrices of order $n_2 \times p$. As a left invariant measure ν on G, we take $d\nu(A,F) = dFdA/|AA'|^{p/2}$ where dA and dF are Lebesgue measures on R^{n_2p} and R^{p^2} respectively. Let $T(X)$ be a maximal invariant under G and denote its distribution under H_1 by $dP_{\sigma_1}^T$ and under H_0 by dP_0^T. Then applying Wijsman's representation theorem (1967), the ration $dp_{\sigma_1}^T/dP_0^T(t(x)) = R_{\sigma_1}(t(x))$ is given by

$$R_{\sigma_1}(t(x)) = \frac{\int_G f(gx|\sigma_1^2,M_2,\Sigma)|AA'|^{(n_1+n_3)/2}d\nu(A,F)}{\int_G f(gx|0,M_2,\Sigma)|AA'|^{(n_1+n_3)/2}d\nu(A,F)} \,. \tag{2.2}$$

The quantity $R_{\sigma_1}(t(x))$ is simplified in the following lemma.

<u>LEMMA 2.1</u>. The ratio $R_{\sigma_1}(t(x))$ in (2.2) is evaluated as

$$R_\eta(t(x)) = (1-\eta)^{n_1/2}|I_p - \eta X_1'X_1(X_1'X_1 + X_3'X_3)^{-1}|^{-(n_1+n_3)/2} \tag{2.3}$$

where $\eta = \sigma_1^2/(1+\sigma_1^2)$.

Proof. The numerator $N_{\sigma_1}(t(x))$ of (2.2) is given by

$$N_{\sigma_1}(t(x)) = |\Sigma|^{-n/2}(1+\sigma_1^2)^{-n_1/2}. \tag{2.4}$$

$$\int_{G\ell(p)}\int_{R^{n_2p}} q(\Sigma^{-1}A(X_1'X_1/(1+\sigma_1^2) + X_3'X_3)A'$$
$$+ \Sigma^{-1}(X_2A'+F-M_2)'(X_2A'+F-M_2))|AA'|^{(n_1+n_3-p)/2}dFdA$$

(using (1.4))

$$= |\Sigma|^{-n/2}(1+\sigma_1^2)^{-n_1/2}.$$

$$\int_{G\ell(p)}\int_{R^{n_2p}} q(\Sigma^{-1/2}A(X_1'X_1/(1+\sigma_1^2) + X_3'X_3)A'\Sigma^{-1/2}$$
$$+ \Sigma^{-1/2}(X_2A'+F-M_2)'(X_2A'+F-M_2)\Sigma^{-1/2})|AA'|^{(n_1+n_3-p)/2}dFdA$$
$$= |\Sigma|^{-(n_1+n_3)/2}(1+\sigma_1^2)^{-n_1/2}.$$

$$\int_{G\ell(p)}\tilde{q}(\Sigma^{-1/2}A(X_1'X_1/(1+\sigma_1^2) + X_3'X_3)A'\Sigma^{-1/2})|AA'|^{(n_1+n_3-p)/2}dA$$

$$= (1+\sigma_1^2)^{-n_1/2}|X_1'X_1/(1+\sigma_1^2) + X_3'X_3|^{-(n_1+n_3)/2}\int_{G\ell(p)}\tilde{q}(AA')|AA'|^{(n_1+n_3-p)/2}dA$$

where $\tilde{q}(V) = \int_{R^{n_2p}} q(V+F'F)dF.$

Since the denominator of (2.2) corresponds to $N_{\sigma_1}(t(x))$ with $\sigma_1 = 0$, the result follows upon simplification.

Remark 2.1. Since the ratio $dP^T_{\sigma_1}/dP^T_0(t(x))$ is independent of q, it follows that any null robust test is also nonnull robust (vide Kariya and Sinha (1985)). In particular the optimum invariant test derived below is null and hence nonnull robust.

If $p = 1$ or $n_1 = 1$, the ratio $R_\eta(t(x))$ is evidently monotone increasing in $\mathrm{tr}\, X_1(X_1'X_1 + X_3'X_3)^{-1}X_1'$, the Pillai's trace statistic, which is the familiar F-statistic for $p = 1$ and Hotelling's T^2-statistic for $n_1 = 1$. Its null robustness for arbitrary p and n_1, under the model (1.2), follows from Kariya (1981). This proves the following result.

THEOREM 2.1. When $\min(n_1,p) = 1$, for testing $H_0: \sigma_1^2 = 0$ versus $H_1: \sigma_1^2 > 0$ under the model (1.2), the test which rejects H_0 for large values of $\mathrm{tr}\, X_1(X_1'X_1 + X_3'X_3)^{-1}X_1'$ is UMPI, whatever be $q \in Q$. The test is null, nonnull and optimality robust.

If $\min(n_1,p) > 1$, no UMPI test exists. But a Taylor series expansion of $R_\eta(t(x))$ with respect to η around $\eta = 0$, coupled with the observation that

$$\sup_{x_1,x_3} \| x_1'x_1(x_1'x_1 + x_3'x_3)^{-1} \| < 1$$

where $\|\cdot\|$ denotes the Euclidean norm, yields

$$R_\eta(t(x)) = 1 + \eta\{K + \mathrm{tr}\, X_1(X_1'X_1 + X_3'X_3)^{-1}X_1'\} + o(\eta) \qquad (2.5)$$

where K is a constant. For an invariant test $\phi(t)$ of size α, its local power is then evaluated as

$$\int \phi(t(x))dP_\eta^T(t(x)) \qquad (2.6)$$

$$= \alpha + \eta\int \phi\{K + \mathrm{tr}\, X_1(X_1'X_1 + X_3'X_3)^{-1}X_1'\}dP_0^T(t(x)) + o(\eta).$$

An application of the Neyman-Pearson lemma gives the following result.

THEOREM 2.2. When $\min(n_1,p) > 1$, for testing $H_0: \sigma_1^2 = 0$ versus $H_1: \sigma_1^2 > 0$ under the model (1.2), the test which rejects H_0 for large values of Pillai's trace statistic $\mathrm{tr}\, X_1(X_1'X_1 + X_3'X_3)^{-1}X_1'$ is LBI, whatever be $q \in Q$. The test is null, nonnull and optimality robust.

Remark 2.2. Under the above setup when $\min(n_1, p) > 1$, if $A \in G_T$ is used in (2.1) where G_T is the group of pxp nonsingular lower triangular matrices with positive diagonal elements, it is not difficult to show that the corresponding ratio $r_\eta(t(x))$ takes the form

$$r_\eta(t(x)) = 1 + \eta\{K_1 + K_2\delta(x)\} + o_x(\eta) \tag{2.7}$$

where K_1, K_2 (> 0) are constants, $o_x(\eta)$ is $o(\eta)$ uniformly in x and

$$\delta(x) = \sum_{i=1}^{p-1} \mathrm{tr}\ X_{1[i]}(X'_{1[i]}X_{1[i]} + X'_{3[i]}X_{3[i]})^{-1}X'_{1[i]} \tag{2.8}$$

$$+ \left(\frac{n_1+n_3-p+1}{2}\right)\mathrm{tr}\ X_1(X'_1X_1 + X'_3X_3)^{-1}X'_1$$

where $X_{j[i]}$ is the nxi submatrix of the nxp matrix X_j consisting of the first i columns of X_j, j = 1, 3. It, therefore, follows that the test which rejects H_0 for large values of $\delta(x)$ is LBI under this group. However, as noted in the Introduction, this test suffers from a serious drawback due to its asymmetry in the use of the p columns of X_1 and X_3.

Going back to the other model (1.5), we note that under the null hypothesis H_0: $\sigma_1^2 = 0$, $T_1 = X'_1X_1 + X'_3X_3$ and $T_2 = X'_2X_2$ are sufficient for the nuisance parameters Σ and σ_2^2. However, their joint distribution is not complete which can be seen as follows. Assume for simplicity $q(u) = 1/2 \exp(-\mathrm{tr}\ u/2)$. Write $T_1 = ((t_{ij}^{(1)}))$, $T_2 = ((t_{ij}^{(2)}))$. Then $E(t_{11}^{(1)}t_{22}^{(2)} - t_{22}^{(1)}t_{11}^{(2)}) = 0$ but "$t_{11}^{(1)}t_{22}^{(2)} - t_{22}^{(1)}t_{11}^{(2)} = 0$, a.e." does not hold. This lack of completeness leads to the obvious difficulty of constructing an optimum invariant test.

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